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Non-iterative estimation of temperature-dependent thermal conductivity without internal measurements

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Abstract

In this work a direct integration method is proposed to estimate temperature-dependent thermal conductivity in a one-dimensional heat conduction domain without internal measurements. By approximating the spatial temperature distribution in the domain as a third-order polynomial of position and by integrating the heat conduction equation over the spatial and temporal domain, the present method estimates the thermal conductivity directly. Also, this method does not require any prior information on the functional form of the thermal conductivity. Some illustrative examples are examined to verify the proposed approach. The proposed approach may also be useful to make sufficiently accurate initial guesses for sophisticated algorithms usually based on iterative refinement scheme.

Keywords: Inverse heat conduction problem; Temperature-dependent thermal conductivity; Direct integral method

1. Introduction

The estimation of the thermal conductivity from a set of measured temperature data, which is a kind of inverse heat conduction problem (IHCP), is considered in this study. In most practical applications, the thermal conductivity is a function of temperature and as a result the heat conduction equation becomes non-linear. Many methods have been proposed to estimate the temperature-dependent thermal conductivity [1–7]. Most of them have stressed on the development of optimization formulations to find a thermal conductivity minimizing the difference between measured and calculated temperatures at pre-specified spatial and temporal points. These usually require iterative procedures [1–4]. Also, some of them are based on the measurements interior to the domain of interest [2,4]. In some cases, such internal measurements may be undesirable. Using the Kirchhoff transformation that can linearize the heat conduction equation with temperature-dependent thermal conductivity, Lesnic et al. [5] and Kim [6] proposed non-iterative solutions to IHCPs.

This study presents a direct approach to the estimation of temperature-dependent thermal conductivity in a one-dimensional time-dependent domain using temporal measurements only at both ends. The present approach basically adopts the integral approach, which was already used by Huang and Özişik [2] and Kim et al. [7,8]. It should be noted that the present approach has a unique feature against the previous ones. In the previous procedures for the inverse estimation of the thermal conductivity, the functional form of the thermal conductivity was assumed. On the contrary, this approach does not require any assumptions on the functional form of the thermal conductivity.

This study considers a one-dimensional non-linear heat conduction domain with heated and insulated ends. If the temperature distribution in the domain can be approximated as a third-order polynomial and its four time-dependent coefficients can be determined from the imposed heat fluxes and the measured boundary

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Nomenclature

a_i	coefficient of model function in Eq. (10): $i = 0, 1, 2, 3$
C	heat capacity per unit volume
k	thermal conductivity
k_0	reference thermal conductivity
l	slab length
Q	dimensionless heat flux ($= ql/k_0\,\Delta T_{\mathrm{ref}}$)
q	heat flux
T	temperature
T_{i}	initial temperature
t	time
x	coordinate

Greek symbols

ΔT_{ref}	reference temperature difference
θ	relative temperature ($= (T - T_{\mathrm{i}})/\Delta T_{\mathrm{ref}}$)
κ	dimensionless thermal conductivity ($= k/k_0$)
ξ	dimensionless coordinate ($= x/l$)
σ	standard deviation
τ	dimensionless coordinate ($= tk_0/Cl^2$)

Subscripts

L	left wall
R	right wall

temperatures, we will show that the temperature-dependent thermal conductivity can be estimated directly. The proposed approach may also be useful to make sufficiently accurate initial guesses for more sophisticated inverse solution methods mostly based on the iterative schemes.

2. Mathematical model

In a one-dimensional homogeneous heat conduction medium as shown in Fig. 1, let us consider a problem to determine the temperature-dependent thermal conductivity, $k(T)$. It is assumed that the domain is bounded on the left end by a heated wall and on the right by an insulated wall. The heat flux into the left wall is set to a constant, q_{L}. For the case of constant heat capacity per unit volume C, the heat conduction is governed by

$$C\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right], \quad x \in [0, l], \;\; t \in (0, \infty) \tag{1}$$

which is subject to the initial and boundary conditions

$$T(t = 0, x) = T_{\mathrm{i}} = 0, \tag{2}$$

$$q(t, x = 0) = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_{\mathrm{L}} \quad \text{and}$$

$$q(t, x = l) = -k\frac{\partial T}{\partial x}\bigg|_{x=l} = 0. \tag{3}$$

For the simplicity, the initial temperature distribution is assumed to be uniform, T_{i}. Introducing the following variables

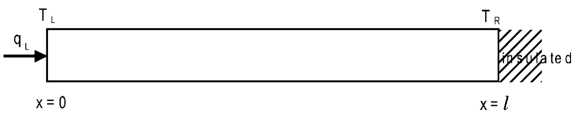

Fig. 1. Description of problem domain.

$$\xi = \frac{x}{l}, \quad \tau = \frac{tk_0}{Cl^2}, \quad \kappa = \frac{k}{k_0},$$

$$\theta = \frac{T - T_{\mathrm{i}}}{\Delta T_{\mathrm{ref}}} \quad \text{and} \quad Q = \frac{ql}{k_0\,\Delta T_{\mathrm{ref}}}, \tag{4}$$

the governing equation and the initial and boundary conditions will be

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial\xi}\left[\kappa(\theta)\frac{\partial\theta}{\partial\xi}\right], \quad \xi \in [0, 1], \;\; \tau \in (0, \infty) \tag{5}$$

and

$$\theta(\tau = 0, \xi) = 0 \tag{6}$$

$$Q(\tau, \xi = 0) = -\kappa\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=0} = Q_{\mathrm{L}} \quad \text{and}$$

$$Q(\tau, \xi = 1) = -\kappa\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=1} = 0. \tag{7}$$

In this, k_0 and ΔT_{ref} are the reference thermal conductivity and the reference temperature difference, respectively. In addition to the above two boundary data (Eq. (7)) it is assumed that two boundary temperatures are available. In practical situations, two boundary heat flux conditions can be imposed and at the same time two boundary temperatures can be measured. The temperatures measured at the ends will be time dependent:

$$\theta(\tau, \xi = 0) = \theta_{\mathrm{L}}(t) \quad \text{and} \quad \theta(\tau, \xi = 1) = \theta_{\mathrm{R}}(t). \tag{8}$$

We are now attempting to determine the temperature-dependent thermal-conductivity non-iteratively only with four boundary data (Eqs. (7) and (8)). For this purpose, we apply the direct integration approach to the present inverse analysis. Integrating Eq. (5) over the spatial domain $[0, 1]$ and over the time interval $[0, \tau]$, we obtain

$$\int_0^1 \theta\, \mathrm{d}\xi = Q_{\mathrm{L}}\tau. \tag{9}$$

The above relation simply represents the relation between the amounts of heat added and the increase in the energy. If the spatial distribution of the temperature is expressed in terms of thermal conductivity as well as other known quantities, Q_L, θ_L and θ_R, we can obtain a relation to estimate the thermal conductivity from the above energy conservation equation. Considering the initial and boundary conditions (Eqs. (6) and (7)) the temperature distribution could be a monotonic function. In this work where since two temperatures and two heat fluxes are available at the both ends, we simply approximate the temperature distribution as a third-order polynomial with time-dependent coefficients [7,8]:

$$\theta(\tau,\xi) = a_0(\tau) + a_1(\tau)\xi + a_2(\tau)\xi^2 + a_3(\tau)\xi^3. \tag{10}$$

The coefficients can be written as

$$a_0 = \theta_L, \quad a_1 = -\frac{Q_L}{\kappa_L},$$
$$a_2 = \frac{2Q_L - 3\kappa_L(\theta_L - \theta_R)}{\kappa_L} \quad \text{and}$$
$$a_3 = -\frac{Q_L - 2\kappa_L(\theta_L - \theta_R)}{\kappa_L}. \tag{11}$$

Of course, κ_L is the non-dimensional thermal conductivity at the temperature of the left end, that is $\kappa_L = \kappa(\theta_L)$. Substituting Eqs. (10) and (11) into Eq. (9), we obtain an expression to determine the temperature-dependent thermal conductivity:

$$\kappa_L = \frac{Q_L}{6(\theta_L + \theta_R) - 12Q_L\tau}. \tag{12}$$

The above simple expression allows us to directly estimate the thermal conductivity at $T = T_L$ only with the imposed and the measured data at both ends. Also, no assumption on the functional form of the thermal conductivity is made to derive the expression.

3. Evaluation of temperature approximation

Since the proposed method should be highly dependent on the approximated temperature profile, it would be worthwhile to appreciate the present temperature approximation of a third-order polynomial. Other temperature distribution functions satisfying four boundary data mentioned above could be possible. However, the basic idea of the present method will not be degraded whatever temperature profiles are introduced. A third-order polynomial could be one of simplest approximates to the temperature distribution and we adopt it.

For the evaluation of the temperature approximation, five examples are introduced. Examples 1 and 2 are related to the thermal conductivity linearly increasing and decreasing, respectively. Examples 3 and 4 deal with quadratic variations and in Example 5 a rather oscillatory functional form of thermal conductivity is considered.

Example 1: $\kappa(\theta) = 0.15 + \theta/300$.
Example 2: $\kappa(\theta) = 0.50 - \theta/300$.
Example 3: $\kappa(\theta) = 0.10 + (\theta - 40)^2/8000$.
Example 4: $\kappa(\theta) = 0.60 - (\theta - 60)^2/10\,000$.
Example 5: $\kappa(\theta) = 0.50 + (\theta/320)\sin(\theta/10)$.

The temperature range of interest is set to from 0 to about 100 °C and the initial temperature and the reference temperature difference are assumed to 0 and 1 °C, respectively. Although some of the examples seem to be rather severe, they would be illustrative in the evaluation of the approximation. A constant heat flux of $Q_L = 5$ is imposed on the left wall at $\tau = 0$ and heating is continued until $\tau = 20$ while the right wall is insulated. If $k_0 = 100$ W/m °C and $C = 4000$ kJ/m^3 °C which are typical for metals, for a specimen of 5 cm thick the heat flux corresponds to $q = 10$ kW/m^2 and the total measuring time will be 2000 s.

The approximated temperature profiles are compared with the exact ones in terms of the relative error defined as

$$\text{Relative error} = \frac{\int_0^1 |\theta_{\text{exact}}(\tau,\xi) - \theta_{\text{approximated}}(\tau,\xi)|\,d\xi}{\int_0^1 \theta_{\text{exact}}(\tau,\xi)\,d\xi}. \tag{13}$$

The exact solutions of the direct heat conduction equation with the above non-linear thermal conductivities are obtained numerically with the finite element method [9]. Relative error for each example is given in Fig. 2 and for the reference the case of a constant thermal conductivity

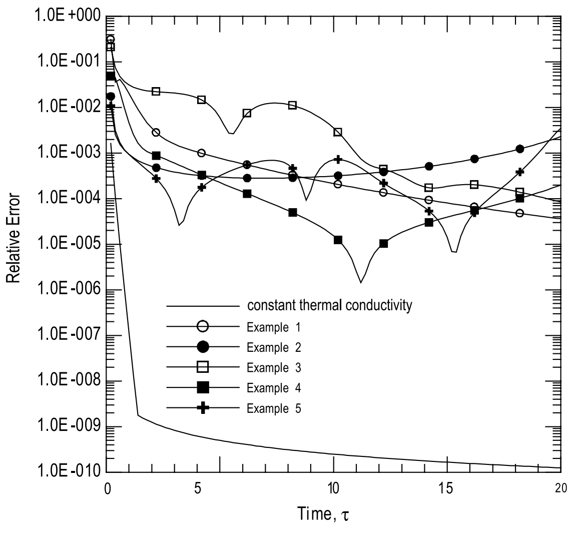

Fig. 2. Relative errors of approximated temperature distribution.

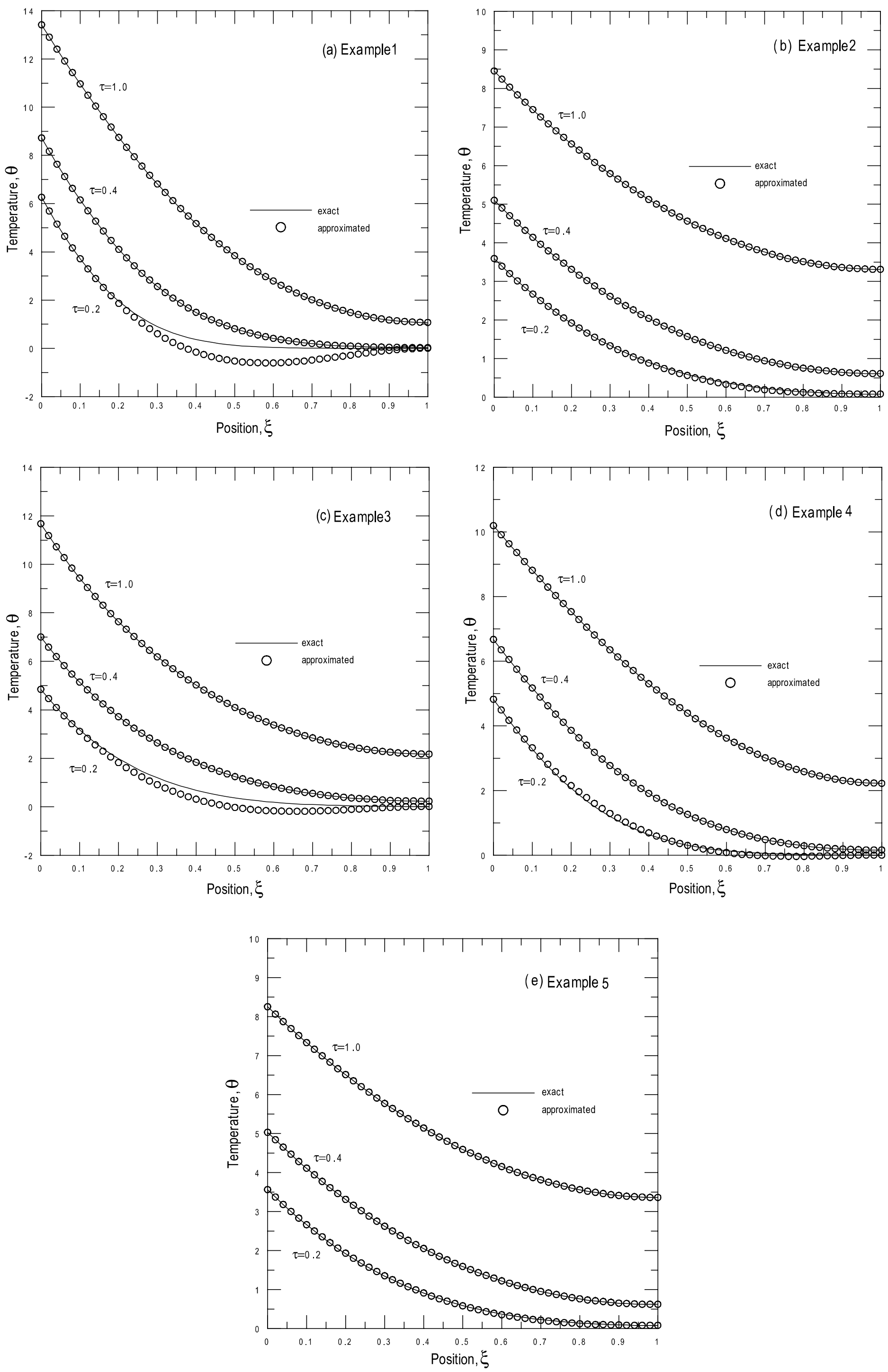

Fig. 3. Temperature profiles for $Q_L = 5$.

($\kappa = 1.0$) is also shown in the figure. In case of a constant thermal conductivity, the agreement between the approximated and the exact profiles are extremely good except for very small τ. During very earlier period when heat imposed from the left wall could not be transferred to the right wall sufficiently, the right wall temperature would remain at the initial temperature and the approximation would be poor. However, relative errors shown in Fig. 2 indicate that for $\tau \gtrsim 1$ the relative error decreases to several percents. As can be seen in Fig. 3, excepting the region of very small τ, the third-order polynomial seems to be a reasonable approximation to the spatial temperature distribution. It should be noted that according to Eq. (12), the error would not be accumulated so that the estimation of the thermal conductivity at the future time step will not be affected by the previous errors.

4. Estimation of thermal conductivity

Now, with the above examples, we will estimate the temperature dependent thermal conductivity. Again, the heat flux is set to $Q_L = 5$ and the temperature measuring time is $\tau = 20$ with a measurement time step $\Delta\tau = 1.0$ (i.e. 20 measurements). Then, the temperature range will be from 0 to about 100 °C when $T_i = 0$ °C and $\Delta T_{ref} = 1$ °C.

In order to verify the effectiveness of the proposed method in real situations, where measurements are usually contaminated by uncontrollable error, the measured temperatures are obtained by adding the random errors of measurement to the exact temperatures

$$\theta_{measured} = \theta_{exact} + \omega\sigma_\theta, \tag{14}$$

where σ_θ is the standard deviation of measurement errors, and for normally distributed errors with 99% confidence bounds ω lies within the bounds $-2.576 < \omega < 2.576$. For the reference, the exact temperature data used in this estimation are listed in Table 1. The estimated results are given in Figs. 4–8. In case of the estimation using the measurement data with error, the error bar denoting the sample standard deviation is plotted to show the statistical behavior in each figure. For the statistical analysis, 100 sets of numerical experiments are conducted. In the numerical experiments, we repeat each IHCP three times with three different error levels: $\sigma_\theta = 0.0$, 0.1, and 0.2. In this, ΔT_{ref} is set to 1 °C so that the standard deviations correspond to 0.0, 0.1 and 0.2 °C. If measurement error is neglected, as can be seen Figs. 4–8, we can predict the temperature-dependent thermal conductivity with excellent accuracy. The estimates with the measurement data contaminated by random error also show that the proposed approach can estimate the thermal conductivity quite reasonably. The results of the statistical analysis suggest that the standard deviation becomes larger with the increase of thermal conductivity. In other words, smaller thermal conductivities result in smaller errors. This observation is the same as the results of Huang and Özişik [2] where the error bounds of the low-diffusivity material were narrower than those of the high-diffusivity material. In fact, this can be inferred from statistics. If the thermal conductivity can be obtained from Eq. (12) and it is a function of θ_L and θ_R only, the standard deviation of the thermal conductivity σ_κ can be obtained with the standard deviation of the temperature σ_θ:

Table 1
Exact boundary temperatures

τ	Example 1		Example 2		Example 3		Example 4		Example 5	
	θ_L	θ_R	θ_L	θ_R	θ_L	θ_R	θ_L	θ_R	θ_L	θ_R
1	13.41	1.07	8.45	3.31	11.67	2.17	10.19	2.22	8.25	3.36
2	18.64	5.36	13.59	8.23	18.33	6.47	14.66	7.47	13.15	8.40
3	23.12	10.60	18.72	13.17	25.05	10.84	19.13	12.81	18.04	13.46
4	27.54	15.95	23.87	18.10	31.92	15.13	23.75	18.05	23.00	18.50
5	32.02	21.26	29.03	23.02	38.65	19.37	28.46	23.22	28.07	23.49
6	36.56	26.54	34.20	27.94	44.86	23.58	33.25	28.34	33.30	28.41
7	41.16	31.77	39.38	32.86	50.34	27.85	38.09	33.43	38.69	33.25
8	45.81	36.97	44.59	37.76	55.14	32.28	42.97	38.50	44.19	38.03
9	50.49	42.15	49.81	42.66	59.37	37.00	47.88	43.55	49.61	42.78
10	55.21	47.31	55.05	47.55	63.22	42.12	52.82	48.58	54.65	47.64
11	59.95	52.45	60.33	52.42	66.81	47.67	57.79	53.60	59.25	52.72
12	64.72	57.57	65.63	57.29	70.30	53.57	62.78	58.61	63.62	58.03
13	69.51	62.69	70.97	62.13	73.83	59.59	67.79	63.61	68.01	63.39
14	74.32	67.79	76.36	66.97	77.49	65.54	72.83	68.59	72.56	68.66
15	79.14	72.89	81.80	71.78	81.34	71.34	77.90	73.56	77.31	73.82
16	83.98	77.97	87.31	76.56	85.38	76.98	82.99	78.52	82.24	78.88
17	88.82	83.05	92.91	81.32	89.60	82.48	88.12	83.47	87.37	83.85
18	93.68	88.13	98.62	86.04	93.95	87.87	93.30	88.39	92.78	88.70
19	98.55	93.20	104.48	90.71	98.42	93.18	98.53	93.29	98.70	93.38
20	103.43	98.26	110.54	95.33	102.98	98.43	103.85	98.15	105.69	97.79

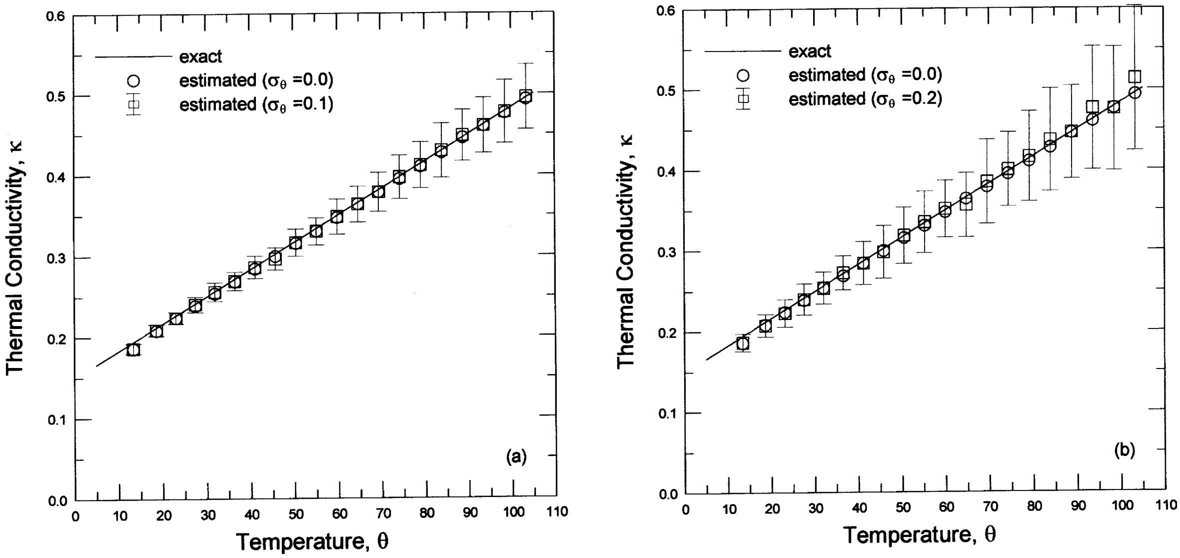

Fig. 4. Estimated thermal conductivity for $Q_L = 5$ (Example 1).

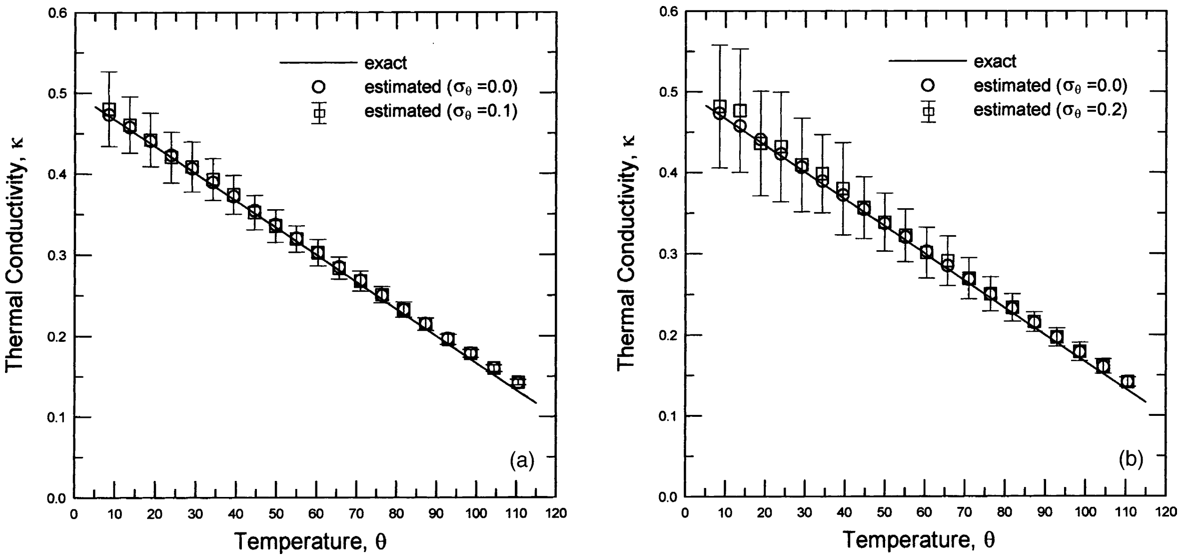

Fig. 5. Estimated thermal conductivity for $Q_L = 5$ (Example 2).

$$\sigma_\kappa = \sqrt{\left(\frac{\partial \kappa_L}{\partial \theta_L}\right)^2 \sigma_{\theta_L}^2 + \left(\frac{\partial \kappa_L}{\partial \theta_R}\right)^2 \sigma_{\theta_R}^2} = \frac{6\sqrt{2}\kappa_L^2}{Q_L}\sigma_\theta \quad \text{or} \quad \frac{\sigma_\kappa}{\kappa_L} = \frac{6\sqrt{2}\kappa_L}{Q_L}\sigma_\theta, \tag{15}$$

where it is assumed that θ_L and θ_R are uncorrelated and their standard deviations are same. The statistical behavior of the thermal conductivity of Eq. (15) obviously indicates the dependency of σ_κ on the magnitude of the thermal conductivity. The equation also says that the increase of heat flux imposed on the right wall will decrease the error bound of the estimated thermal conductivity. As the applied heat flux increases, however, the time taken to reach the prescribed temperature bound, for instance we are now considering the temperature range from 0 to 100 °C, will shorten and small-τ range generating poor approximation of the temperature profile will broaden. The plot of the temperature differences $\theta_L - \theta_R$ in Fig. 9 drops a hint that

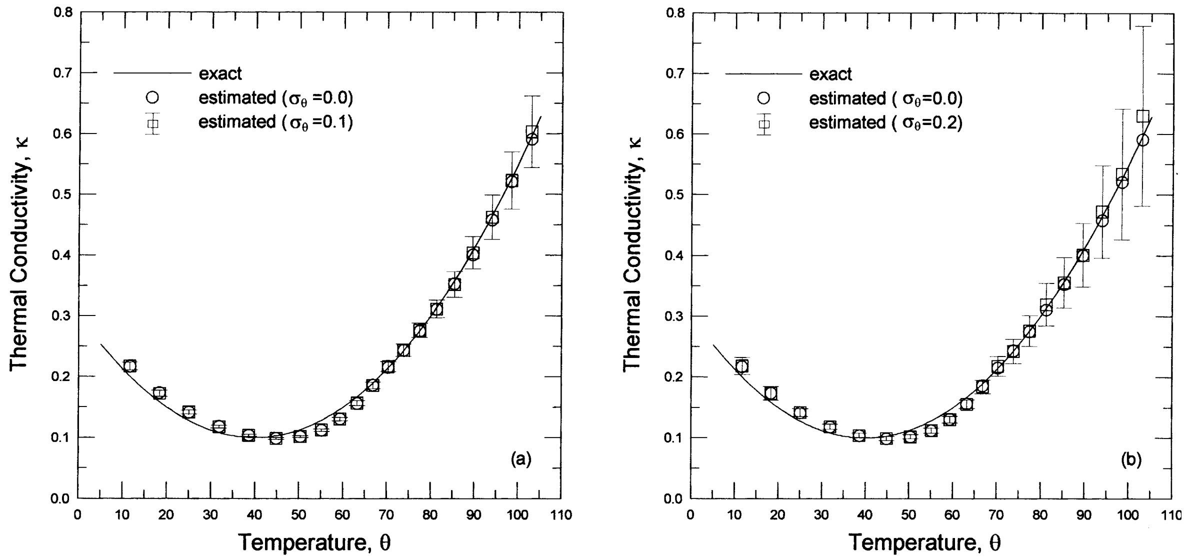

Fig. 6. Estimated thermal conductivity for $Q_L = 5$ (Example 3).

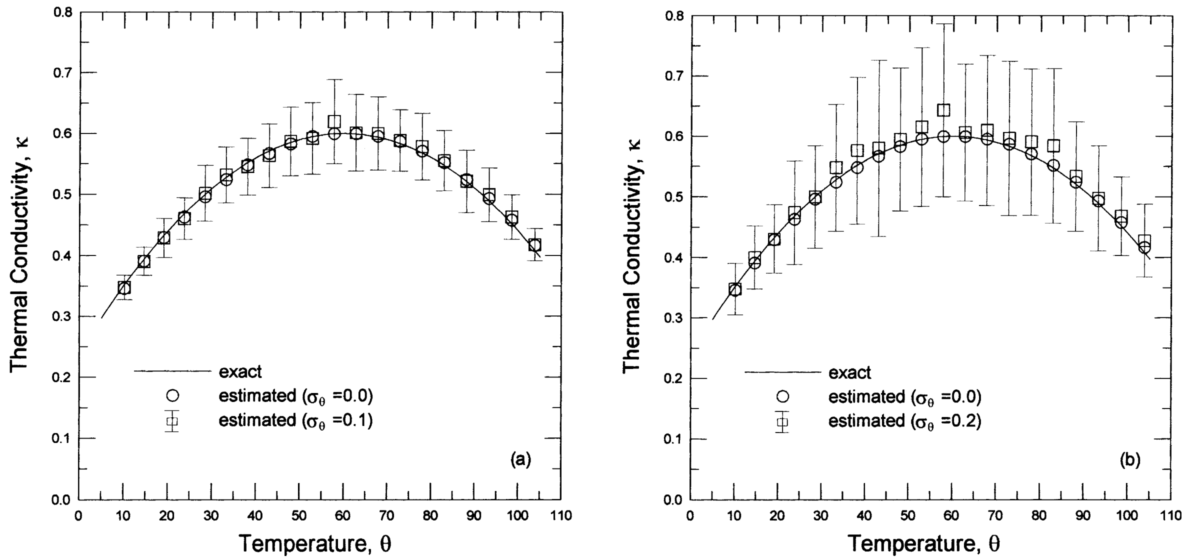

Fig. 7. Estimated thermal conductivity for $Q_L = 5$ (Example 4).

bigger temperature difference tends to cause more error in the temperature approximation (compare Figs. 2 and 9a). According to our experience, it is recommended that the temperature difference should be kept less than about 20 °C when $\Delta T_{ref} = 1$ °C and it appears not to be too restrictive condition in practical situation. If we increase the heat flux to $Q_L = 10$, the standard deviations of the estimated thermal conductivities will be halved even though for some cases committing small-τ or large-$(\theta_L - \theta_R)$ criterion the estimation will be poor. We repeat the estimations of the thermal conductivity with the increased heat flux of $Q_L = 10$ and the measuring time is set to 10 in order that the temperature range of interest remains from 0 to 100 °C. The variations of $\theta_L - \theta_R$ are given in Fig. 9b and the estimated thermal conductivities based on the measured temperatures with a standard deviation of $\sigma_\theta = 0.2$ are shown in Fig. 10. For the statistical analysis, 100 sets of measurements are conducted. As deduced from the statistical analysis in Eq. (15), the standard deviations for the estimates are halved. As expected, the quality of the estimations is degraded where the temperature differences go over $\theta_L - \theta_R = 20$ or the

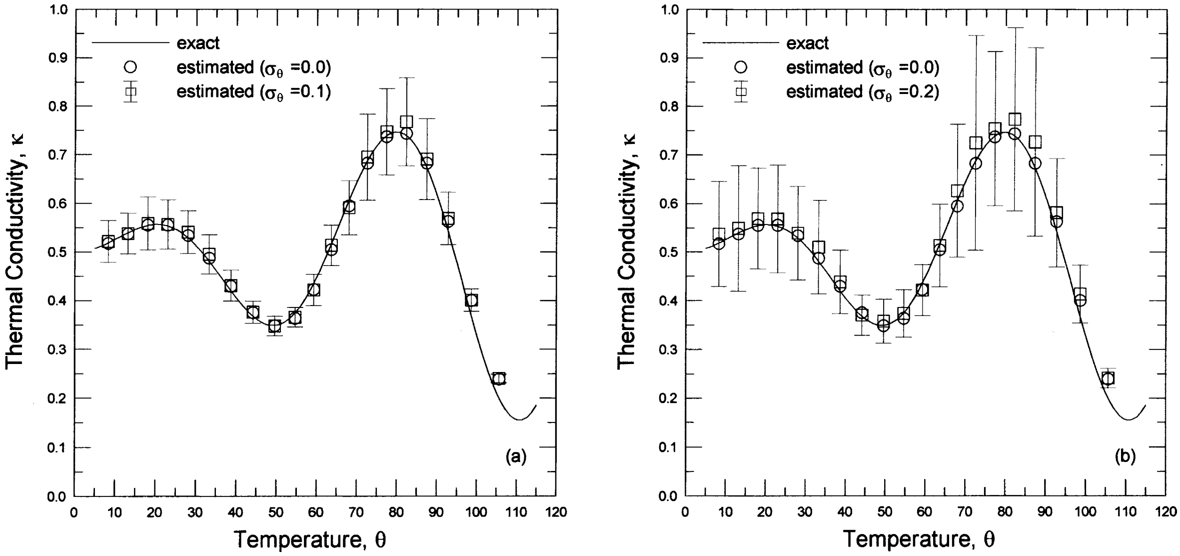

Fig. 8. Estimated thermal conductivity for $Q_L = 5$ (Example 5).

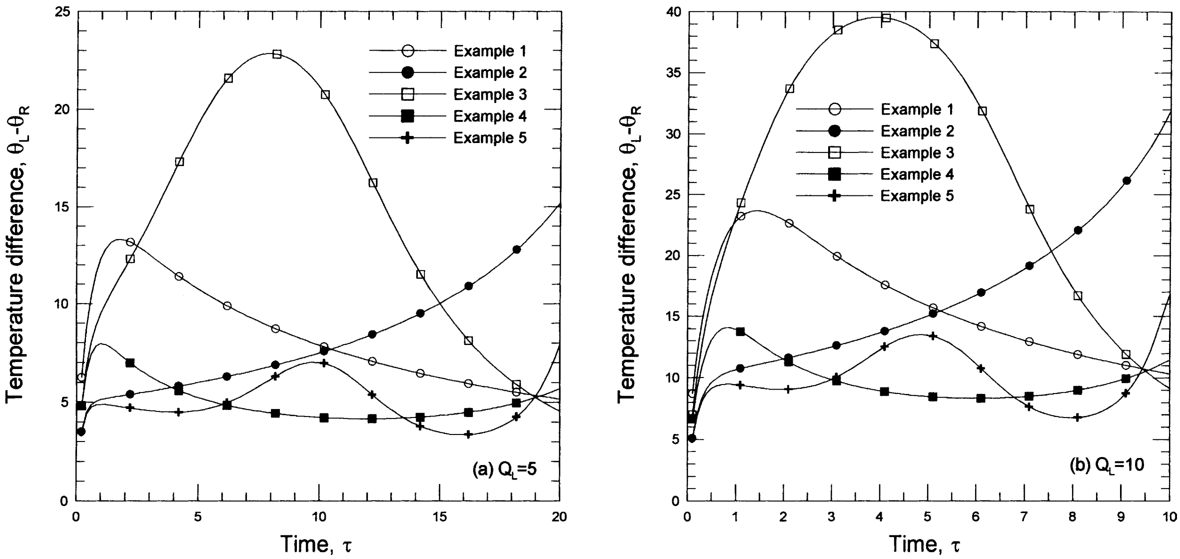

Fig. 9. Temperature differences.

elapsed time is $\tau \lesssim 1$ (see Fig. 10c). However, it is worthwhile to note that the estimate of the thermal conductivity based on the measured temperatures at a certain time is independent of errors engaged in the previous measurements. Namely, in the region of $\theta_L - \theta_R \lesssim 20$ and $\tau \gtrsim 1$, the quality of the estimates would be restored. The time interval between consecutive measurements does not affect the accuracy of the estimates. The proposed algorithm can also be used to generate the initial guesses for more elaborated inverse algorithms usually resorting to iterative methods.

5. Conclusions

The present study proposes a direct approach to estimate the temperature-dependent thermal conductivity in a one-dimensional non-linear heat conduction medium. Although the approach is based on the conventional integral approach, it has some peculiar features that it does not require any assumption on the functional form of the thermal conductivity, any iterative procedures, and any interior temperature sensors. By approximating the temperature distribution in a one-

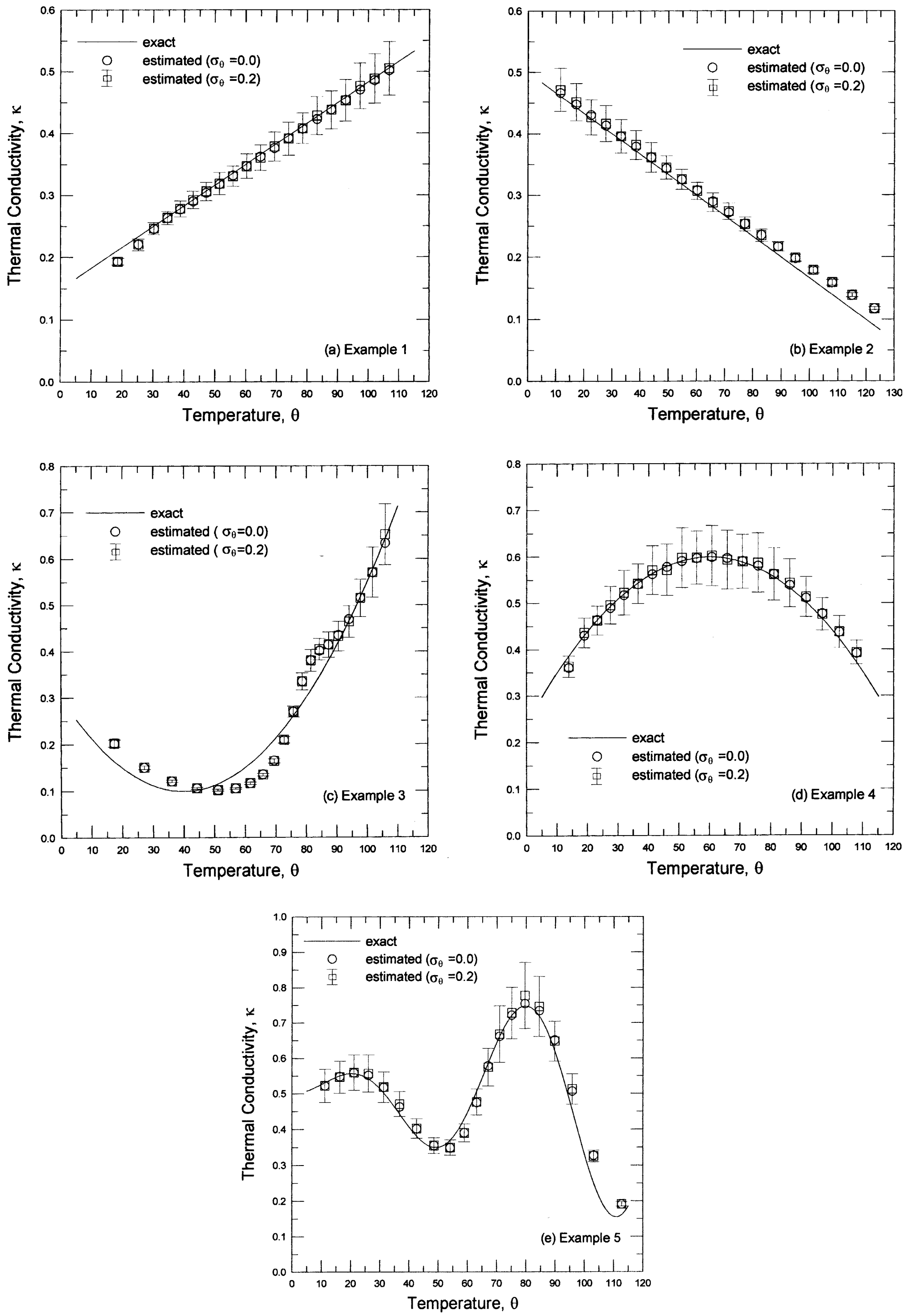

Fig. 10. Estimated thermal conductivity for $Q_L = 10$.

dimensional domain with the heated and the insulated walls as a third-order polynomial, we can derive a simple relation between the thermal conductivity at the heated wall temperature and the known variables like end temperatures, heat flux and time. Several artificial thermal conductivity functions expressed in terms of temperature are successfully estimated with the proposed method. Some restrictions in the application of the proposed algorithm and statistical issues are discussed. Also, the estimates with the present method would be good initial guesses for more sophisticated IHCP algorithms.

Acknowledgements

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References

[1] C.H. Huang, J.Y. Yan, H.T. Chen, Function estimation in predicting temperature-dependent thermal-conductivity without internal measurements, J. Thermophys. Heat Transfer 9 (4) (1995) 667–673.
[2] H.T. Chen, J.Y. Lin, Simultaneous estimations of temperature-dependent thermal conductivity and heat capacity, Int. J. Heat Mass Transfer 41 (14) (1998) 2237–2244.
[3] C.Y. Yang, Estimation of the temperature-dependent thermal conductivity in inverse heat conduction problems, Appl. Math. Modell. 23 (6) (1999) 469–478.
[4] C.H. Huang, M.N. Özişik, Direct integration approach for simultaneously estimating temperature-dependent thermal-conductivity and heat-capacity, Numer. Heat Transfer: Part A 20 (1) (1991) 95–110.
[5] D. Lesnic, L. Elliott, D.B. Ingham, A note of the determination of the thermal properties of a material in a transient nonlinear heat conduction problem, Int. Commun. Heat Mass Transfer 22 (4) (1995) 475–482.
[6] S. Kim, A simple direct estimation of temperature-dependent thermal conductivity with Kirchhoff transformation, Int. Commun. Heat Mass Transfer 28 (4) (2001) 537–544.
[7] S. Kim, M.C. Kim, K.Y. Kim, An integral approach to the inverse estimation of temperature-dependent thermal conductivity without internal measurements, Int. Commun. Heat Mass Transfer 29 (1) (2002) 107–113.
[8] S. Kim, M.C. Kim, K.Y. Kim, A direct integration approach for the estimation of time-dependent boundary heat flux, KSME Int. J. 16 (10) (2002) 1320–1326.
[9] E.B. Becker, G.F. Carey, J.T. Oden, in: Finite Elements, vol. 1, Prentice Hall, Englewood Cliffs, NJ, 1981, pp. 40–130.